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Spectrum of Seven Historical Cases Reveals its True Colors through the Prism of Norton's Dome

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Abstract:

For almost a quarter of a century, Norton dome has stood as a classic weapon of mass deterrence against Newtonian determinism. Because of its non-Lipschitz nature, it involves a rigid, non-quantum ball sliding spontaneously along a smooth surface, without apparent reason and in violation of the principle of inertia. This work proposes a new solution to the Norton dome problem, which consists of integrating the differential equations of dynamics over a sufficiently large portion of the dome using generalized coordinates and deducing a formal contradiction in both the physical and mathematical sense: if any motion occurs, the ball becomes ubiquitous, moving simultaneously in different directions. The inertia principle can then be demonstrated as a Cauchy-Lipschitz theorem generalized to non-Lipschitz systems, whether they are in linear motion or rotating. This ubiquity method is then applied in detail and for the first time to several similar historical non-Lipschitz cases (from scholars such as Poisson, Duhamel, Boussinesq, Bertrand…) that sparked lively scientific and philosophical debates in the 19th century concerning determinism and the existence of free will in the physical universe. This questioning attracted attention again in the 21st century with Norton's dome. In light of one's solution to the latter, these cases themselves appear to be purely contradictory rather than exceptions or threats to the basic principles of physical science. Furthermore, by interpreting the inertia principle as a law of irreversibility, one demonstrates the original historical emergence of a non-statistical fundamental thermodynamics, endowed with a true atomic arrow of time.

Keywords: *Indeterminism; Inertia; Non-Lipschitzian; Norton Dome; Thermodynamics; Ubiquity.*

طيفٌ من سبع حالات تاريخية يكشف عن حقيقته من خلال منظور قبة نورتون

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ملخص:

على مدى ربع قرن تقريبًا، مثّلت قبة نورتون سلاحًا كلاسيكيًا للردع الجماعي ضد الحتمية النيوتونية. نظرًا لطبيعتها غير الليبشيتزية، فإنها تتضمن كرة صلبة غير كمومية تنزلق تلقائيًا على سطح أملس، دون سبب واضح، في انتهاك لمبدأ القصور الذاتي. يقترح هذا العمل حلًا جديدًا لمشكلة قبة نورتون، يتمثل في تكامل المعادلات التفاضلية للديناميكا على جزء كبير بما فيه الكفاية من القبة باستخدام إحداثيات معممة، واستنتاج تناقض رسمي بالمعنى الفيزيائي والرياضي: إذا حدثت أي حركة، تصبح الكرة منتشرة في كل مكان، وتتحرك في آن واحد في اتجاهات مختلفة. يمكن بعد ذلك إثبات مبدأ القصور الذاتي كنظرية كوشي–ليبشيتز معممة على الأنظمة غير الليبشيتزية. ثم تُطبَّق هذه الطريقة الشاملة بالتفصيل، ولأول مرة، على عدة حالات تاريخية مماثلة لا تخضع لنظرية ليبشيتز (من علماء مثل بواسون، ودوهامل، وبوسينسك، وبرتراند، وغيرهم) والتي أثارت نقاشات علمية وفلسفية حيوية في جميع أنحاء العالم في القرن التاسع عشر، ولا تزال مستمرة حتى يومنا هذا، حول الحتمية، واللاحتمية، ووجود الإرادة الحرة في الكون المادي. في ضوء حل قبة نورتون، تبدو هذه الحالات نفسها متناقضة تمامًا وليست استثناءات أو تهديدات للمبادئ الأساسية للعلوم الفيزيائية. علاوة على ذلك، من خلال تفسير مبدأ القصور الذاتي كقانون للانعكاسية، يُبرهن على الظهور التاريخي الأصلي للديناميكا الحرارية الأساسية غير الإحصائية، المزودة بسهم زمني ذري حقيقي.

الكلمات المفتاحية: *اللاحتمية؛ القصور الذاتي؛ اللاليبشيتزية؛ قبة نورتون؛ الديناميكا الحرارية؛ الانتشار في كل مكان.*

1. Introduction

First of all, explanation of the topic in the light of the current literature should be made in clear, and precise terms as if the reader is completely ignorant of the subject. In this section, establishment of a warm rapport between the reader, and the manuscript is aimed. Updated, and robust information should be presented in the 'Introduction' section**.**

From the 19th century onward, renowned scientists, beginning with Poisson in 1806 (van Strien, 2014), debated the universality of Newtonian determinism. Indeed, the study of certain differential equations of dynamics seemed to reveal a multiplicity of solutions where determinism imposed a single behavior on a body moving in a force field, depending on the given initial conditions. It was only then that a general mathematical result clarified the question of the existence and uniqueness of solutions to differential equations, based on their conformity to a particular criterion, known as the Lipschitz criterion (Lipschitz, 1876).

For the comfort of the novice reader on the subject, one will start from the basics with the main result on Lipschitzian differential equations.

The Cauchy-Lipschitz/Picard–Lindelöf Theorem

A) Statement (first-order ODE)

Consider the differential equation (1):

$$\frac{dy}{dt} = f(t\,, y) \qquad (1)$$

With the initial condition $\boldsymbol{y(t_0) = y_0}$, where $\boldsymbol{f}$ is defined on a domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ and takes values in $\mathbb{R}^n$.

Assumptions:

There exists a neighborhood of (t_0, y_0) such that:

- $f(t, y)$ is continuous;
- $f(t, y)$ is Lipschitz continuous with respect to y, that is:

  $\| f(t\,, y_1) - f(t\,, y_2) \| \leq L \cdot \| y_1 - y_2 \|$,

  for all y_1, y_2 in a neighborhood of y_0, and for some constant $L > 0$.

Conclusion:

Then there exists a time interval $I = [t_0 - \varepsilon, t_0 + \varepsilon]$, with $\varepsilon > 0$, and a unique solution $y(t)$ defined on I, of class C^1, such that:

$$dy/dt = f\big(t, y(t)\big), \text{ with } \boldsymbol{y(t_0) = y_0}.$$

B) Extension to Second-Order ODEs

Consider the second-order equation (2):

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \qquad (2)$$

Let one define a new variable $v(t) = dy/dt$. Then one rewrites (2) as the system (3):

$$dy/dt = v \qquad (3)$$
$$dv/dt = f(t, y, v)$$

This is a first-order system in two variables:

$$dY/dt = F(t, Y), \text{ where } Y = (y, v)$$

Theorem:

If $f(t, y, v)$ is continuous and Lipschitz continuous in (y, v), then the system has a unique solution near the initial condition.

C) Special case: autonomous second-order ODE

Suppose the equation is autonomous, i.e. time-independent like in (4):

$$\frac{d^2y}{dt^2} = f(y) \qquad (4)$$

Then define $v = dy/dt$. The equation (4) becomes the system (5):

$$dy/dt = v, \mathrm{d}v/\mathrm{dt} = f(y) \qquad (5)$$

If $f(y)$ is Lipschitz near the initial condition y_0, then the system has a unique solution.

Usually, "anomalous" solutions appeared only in extreme situations involving imaginary forces, systems of infinite mass, extraterrestrial invaders, etc. (Bhat & Bernstein, 1997). But at the beginning of the 21st century, a groundbreaking article (J. Norton, 2003, 2008) presented the indeterminacy of a simple ball in equilibrium at the top of a dome of a particular shape in a classical gravitational field.

Norton illustrates this dome with Figure 1 below:

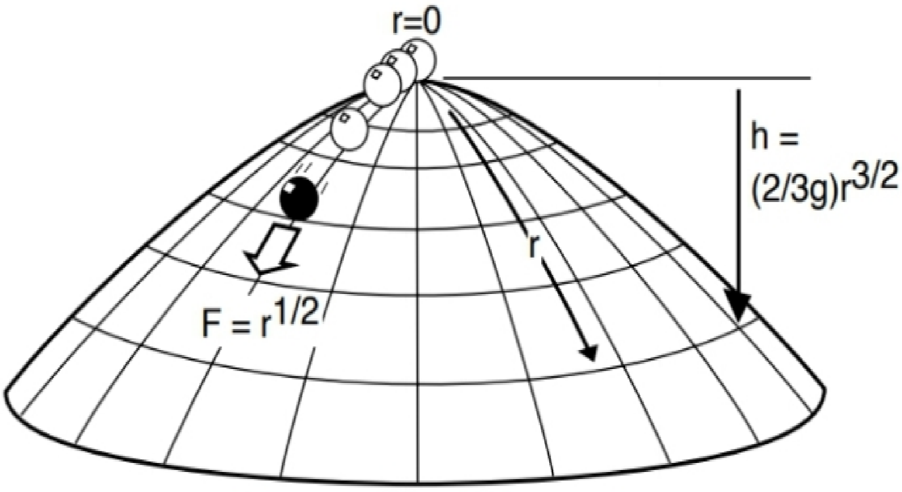

Figure 1: Norton's dome (Norton, 2003)

However, this dome does not satisfy the conditions of the Cauchy-Lipschitz (Picard-Lindelöf) theorem on the uniqueness of solutions to differential equations of the type (6):

$$d^2r/dt^2 = f\big(r(t)\big) \qquad (6)$$

with initial conditions $r(0) = r_0$ and $dr/dt(0) = v_0$.

On Norton's dome, without friction force, the dynamics equation (7) (Norton, 2003):

$$\frac{d^2r}{dt^2} = \sqrt{r} \qquad (7)$$

Has not its Lipschitz condition satisfied, given that the net force on the right-hand side in square root of the curvilinear abscissa r is not differentiable at zero with respect to r. Indeed, $f(r) = \sqrt{r}$ is not Lipschitzian in $r = 0$ since (8):

$$lim_{r\to 0} \frac{(\sqrt{r})}{r} = \infty \qquad (8)$$

⇒ The theorem does not apply.

This leads to an infinite number of possible solutions: the point mass at the top seems to start without cause and slide, under the sole effect of its weight, following a random direction and at an unpredictable instant, along the wall of the dome with the equation (9):

$$h = \left(\frac{2}{3g}\right) r^{3/2} \qquad (9)$$

In the polar coordinate system attached to the point, the components of weight vector P are precisely (10)-(11):

$$P_r = g.\sin\theta \qquad (10)$$
$$P_\theta = g.\cos\theta \qquad (11)$$

Where $\boldsymbol{\theta}$ is the angle between the tangent to the dome at a given point and the horizontal axis $\boldsymbol{x}$. One gets the following relations (12)-(14) (Malament, 2008):

$$\sin\theta = \frac{dh}{dr} = \frac{\sqrt{r}}{g} \qquad (12)$$

$$P_r = \sqrt{r} \qquad (13)$$

$$P_\theta = \sqrt{g^2 - r} \qquad (14)$$

From which the above trajectory equation (7) of the mass located by its curvilinear coordinate r is deduced:

$$\frac{d^2r}{dt^2} = \sqrt{r}$$

Two types of solutions to one's differential equations are then to consider. One is the classical solution (15) of permanent rest for all $\boldsymbol{t}$ of the mass at the apex:

$$\forall t, r(t) = 0 \qquad (15)$$

The other solutions (16)-(17) are the following:

$$\forall T \geq 0,$$

$$If\ t < T:\ r(t) = 0 \qquad (16)$$

$$If\ t \geq T:\ r(t) = \frac{1}{144}(t - T)^4 \qquad (17)$$

It would appear that a ball initially at rest and in perfect equilibrium between its weight and the reaction force of its support, which leaves its highest point at an arbitrary instant T and begins to slide without any external physical intervention, is in flagrant violation of the principle of causality (no initial net force), the principle of inertia (any mass at rest or in uniform rectilinear motion retains its state as long as no external resultant force acts upon it), and determinism (the same initial state must lead to only one possible trajectory).

Moreover, by following a random direction, each particular solution breaks Curie's principle (Curie, 1894), according to which effects have at least the symmetries of their causes. Only the superposition of all possible trajectories starting at the same instant around the dome restores its cylindrical symmetry.

Norton accepts a causality and indeterminism of the situation, but for him, the principle of inertia would be respected because no force is exerted on the ball at the "instant of excitation" $t = T$, and, apart from this instant, there is no initial instant at which the motion is not accompanied by a force.

First, it is not entirely certain that no cause is at play at time zero. Indeed, a constant appears in the fourth derivative (18) of the position with respect to time, or second derivative of the force:

$$\frac{d^4r}{dt^4} = \frac{1}{6} \qquad (18)$$

This phenomenon, unrelated to Newton's principle of the complete determination of a trajectory from its initial position and velocity, seems to occur at the level of the "acceleration" of the resultant force at $t = T$ (called the *jounce*), which will then influence, through integration, the force itself, then the velocity, and finally the position from the following instant $T^+ = T + dT$.

Furthermore, in the case of the dome, it must be emphasized that at no point in the experiment does a force other than its weight and the support act upon the object: the ball's motion results solely from the forces already present at the initial instant, which spontaneously become unbalanced, without any external disturbance, by significantly altering their initial angle along the dome, hence its spontaneous nature by definition.

Finally, simply being in perfect agreement with the second law of motion is not enough to respect the first; new forces, different from the initial forces at equilibrium, must also come into play. The phenomenon of spontaneity is precisely the ability of a system to move on its own using its own initial forces, like Baron Munchausen freeing himself and his horse from the swamp by pulling on his own ponytail. In this sense, there would indeed be a contradiction with the principle of inertia, whose corresponding solution is $r(t) = 0$ for all t.

However, knowing that all of Norton's solutions are derived from solving the same Newtonian equation, it becomes difficult to determine which one to reject as non-physical. A kind of impartial arbiter, external to strict Newtonian physics, would be needed to decide between these different solutions.

2. Resolution of Norton's dome by ubiquity method

2.1 On a symmetrical profile

The ball is now constrained to a specific section of the dome, and its motion is studied along this complete profile, excluding all others. To avoid imposing left or right displacement on this plane, negative values will be assigned to the curvilinear coordinate, which we will call s (zero at the apex of the dome). A polar coordinate system (u_r, u_θ) will be chosen that is suitable for this relative coordinate s. The curve of the dome will have the following equation (19):

$$h = \left(\frac{2}{3g}\right)|s|^{\frac{3}{2}} \qquad (19)$$

Considering each half-profile, one can then immediately get the new equation (20) of motion on the complete dome profile:

$$\frac{d^2|s|}{dt^2} = \sqrt{|s|} \qquad (20)$$

Which amounts of finding a C^2 positive function of time. The search for solutions to differential equations relies on the desire to find functions that are perfectly defined in the mathematical sense, i.e., where each preimage t induces at most one unique image $f(t)$ under its function f. This is a way of rejecting any ubiquity of solutions, in the sense of multi-valued and therefore mathematically improper relationships between two sets. In addition to the permanent rest solution (15), the following non-trivial solutions (21)-(22) are derived from Norton's (16)-(17):

$$\forall T \geq 0,$$

$$For\ t < T: \ \ s(t) = 0 \qquad (21)$$

$$For\ t \geq T: \ \ |s(t)| = \frac{1}{144}(t-T)^4 \qquad (22)$$

(22) $\Rightarrow \ s(t) = \frac{1}{144}(t-T)^4$ on the semi-axis $s > 0$

(22) $\Rightarrow \ s(t) = -\frac{1}{144}(t-T)^4$ on the semi-axis $s < 0$

This different approach, involving the absolute value of s, is then based on a *double resolution*: first, find a positive solution r^+ of the differential equation of dynamics on a half-profile, then extract s from its absolute value in $|s| = r^+$. These are purely mathematical consequences of the equations involved and of their unknown physical quantity expressed as an absolute value. No physical principles are required either (principle of symmetry, inertia, causality...).

Seeking a general solution s for the complete profile is an important precaution to avoid prejudging the physical direction the moving object will take on this particular plane. Figure 2 shows the time-parametric representation of these solutions, illustrating the simultaneous displacement of the mass on either side of the dome's profile:

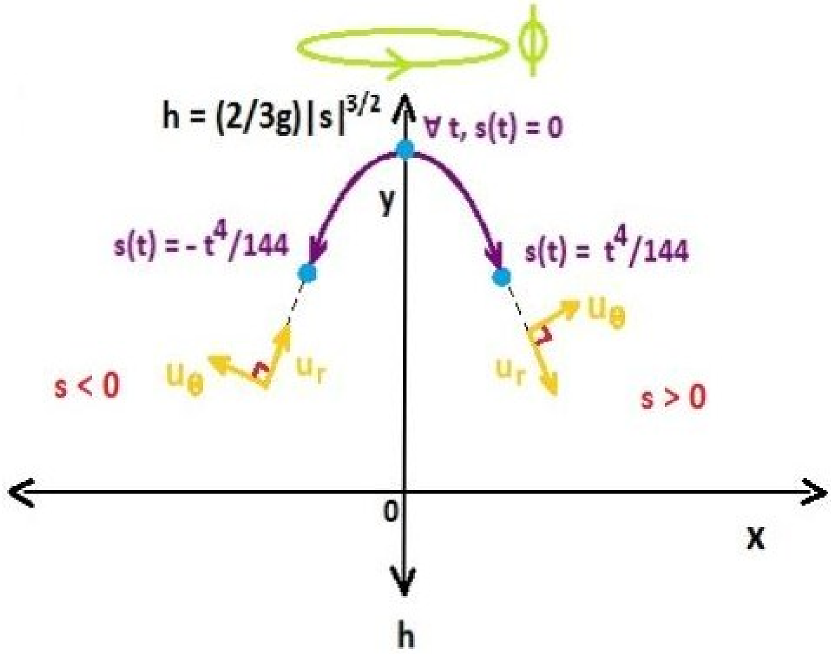

Figure 2: symmetrical ubiquity of the ball on a full profile of the dome

In this section of the dome study, the particle obeys the principle of symmetry in its strongest sense, that of ubiquity rather than an alternative multiplicity of solutions to the equation of motion. The non-trivial Norton solutions for $s \geq 0$ merely provide a window obscuring the overall view of the set of solutions $s(t)$ and their contradictions on the dome's profile. Here, there is neither probability nor an arbitrary choice between the two directions, but a simple classical contradiction.

Similarly, solving the equation $x^2 = 1$ yields the alternative values $x = 1$ or $x = -1$ if one considers only a single half-plane along the x-axis, whereas the complete map of these solutions shows their simultaneous nature: there is clearly superposition, not alternation, of the branches of the curve $y = x^2$ across the entire graphic plan. It's like an observer moving around the vertical axis y and solving the equation from different points of view. The following figure 3 is particularly telling:

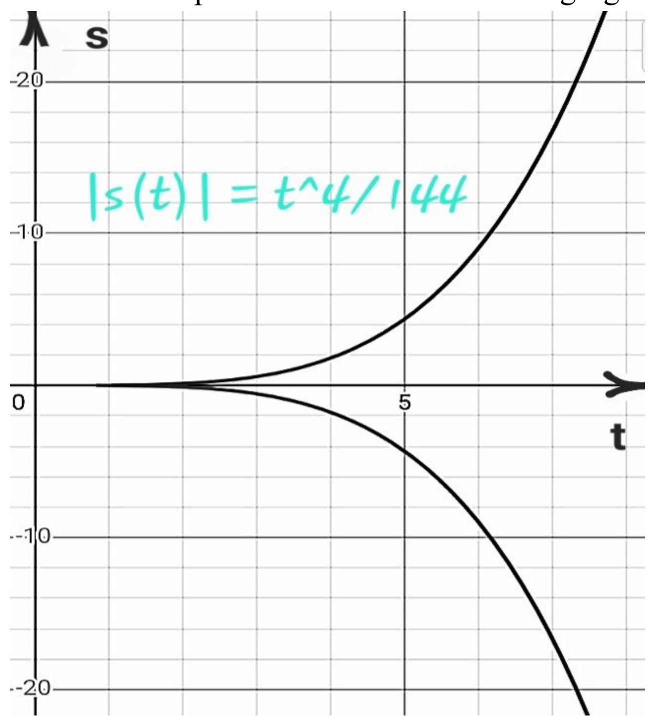

Figure 3: graphical representation of an absolute value function along a complete y-axis

Since this ubiquity is repeated all around the dome's axis of rotation, all non-trivial Norton solutions—i.e., those other than the ball remaining at constant rest—are ultimately physically and mathematically contradictory, regardless of the plane of study considered. Even if, by chance, an arbitrary “hyperjerk” were to create a transverse force displacing the particle toward a physical section other than the one under study, the same ubiquitous behavior would be reproduced on this new section.

Besides, nothing in our proof requires to choose the curvilinear abscissa s on the same plane: thanks to the perfect cylindrical symmetry of the force fields around the axis of rotation, the arcs with positive and negative values can belong respectively to two different planes of study forming any angle ϕ, making s a function of time and ϕ, i.e., $s = s(t, \phi)$, which also obeys a differential equation in absolute value $|s|$.

Importantly, the preceding demonstration is not limited to solutions derived by Norton but can be identically transposed to any other non-zero mathematical solution of the motion $r^*(t)$ defined on the positive half-profile of the dome and over any time interval I. To show the physical and mathematical impossibility of these new solutions, it suffices to solve the fundamental differential equation of dynamics with $|s| = r^*$ on a complete profile, hence, respectively, for all $t \in I$, $s(t) = r^*(t)$ and $s(t) = -r^*(t)$ on either side of the profile under consideration.

The method works even for a trajectory of r^* with a variable sign, by isolating the purely positive or negative parts. As above, it is not necessary to choose two coplanar half-profiles: the inconsistency of the ball's motion can also be demonstrated on a simple portion (“slice of cake”) of the Norton dome with angle ϕ by defining the curvilinear abscissa s on different sections hinged around the axis of symmetry.

It should be noted here that the dynamic equation obtained with the absolute value of the target variable remains unchanged regardless of the initial conditions of the problem: indeed, even in the cases $s(0) \neq 0$ or $ds/dt \neq 0$, solving $|s|$ will always yield two branches for s, but one of them will necessarily be eliminated by the non-zero initial conditions. This breaking of symmetry in the plane of study would prohibit the existence of a solution branch of the motion on one of the half-profiles, thus eliminating the ubiquity automatically generated in the system. Such mathematical elimination is no longer possible if all initial conditions are zero.

From all this, one can conclude that, in accordance with the principle of inertia, the state of continuous rest is the only mathematically well-defined and physically valid solution to the Norton’s dome problem and an infinite number of other non-Lipschitzian related cases.

2.2 On an asymmetric half-profile

Yet, on an isolated half-profile of the dome (corresponding to a portion angle $\phi = 0$), without resorting to a symmetry of the force field over a larger space, things are different. One needs to see "what happens" on the left side of the half-profile both to justify why the mass is “constrained” to remain on this right half-plane and to detect any paradox capable of dismissing violations of the inertia principle. One would like to find the same kind of ubiquity between, for example, the rest on one side (the "platform" of the dome on the right) and the motion of the ball on the dome on the other.

Nevertheless, given the non-Lipschitz nature of the forces at the dome's apex, nothing should prevent the mass from rolling down the dome's slope, thus following a spontaneous trajectory compatible with the solutions for the left half-plane, where the ball would be stationary at the apex at $t = T$ and then "disappear" at $t > T$.

Even if the force field on the left were chosen to be Lipschitz-like to guarantee the uniqueness of the solution on this half-plane (and again create a contradiction with the behavior of the moving object on the right), the Cauchy-Lipschitz theorem would not be applicable if the force field is not defined over a whole neighborhood of points, that is, a set containing an open interval of the initial conditions. However, the existence of a unique solution on a half-profile can be demonstrated by means other than Cauchy-Lipschitz, such as elementary integration.

Now, one is able to prevent by ubiquity Norton's spontaneous solutions in asymmetrical configurations of the dome, with no longer having two perfectly superimposable half-profiles. Let one take for example the ball at rest at $t = 0$ ($x_0 = 0, h_0 = 0$) at the edge of the dome on the right, with a precipice, a wall, or a plane–all net force fields being tangentially zero–on the left (see *figure 4*):

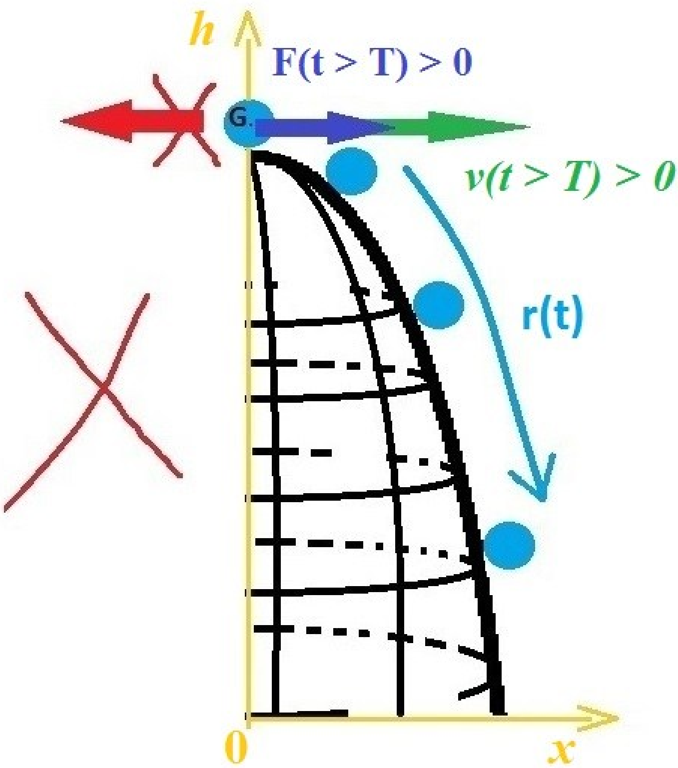

Figure 4: asymmetrical ubiquity of the ball on a half-profile

One then solves the fundamental principle of dynamics on each side, wondering what global physical movement the ball would follow on this half-plane of study. On the right, solutions would be the ball starting to spontaneously descend the wall of the dome at any time or the ball staying continuously at rest. On the left, the fundamental principle gives:

- On the negative $\boldsymbol{x}$-axis: $d^2x(t)/dt^2 = 0$, thus, after integration, the unique solution, $\forall t$: $x(t) = 0$. No hyper-jerk is involved here.
- On the h axis: $\forall t$, $h(t) = 0$, since the ball doesn't flip to the left.

By bringing together all behaviors on both sides for the same ball, the ubiquity paradox still arises that if the object starts towards the right, it would remain at rest at the same time on the left. Then, it appears that on the right side the only solution compatible with one's initial rest conditions would be (23)-(24) for all $\boldsymbol{t}$:

$$x(t) = 0 \qquad (23)$$
$$h(t) = 0 \qquad (24)$$

i.e. the ball standing rest at the apex.

Let one finally mention that Norton proposes another way to obtain his "acausal solutions": he asks to consider a mobile starting from the bottom of the dome to which one would impart an energy or initial speed sufficiently calibrated to hoist it exactly to the top (Figure 5 below). If one reverses the movement one would find, by the well-known principle of time invariance of Newtonian differential equations, the spontaneous sliding movement of the mass in question:

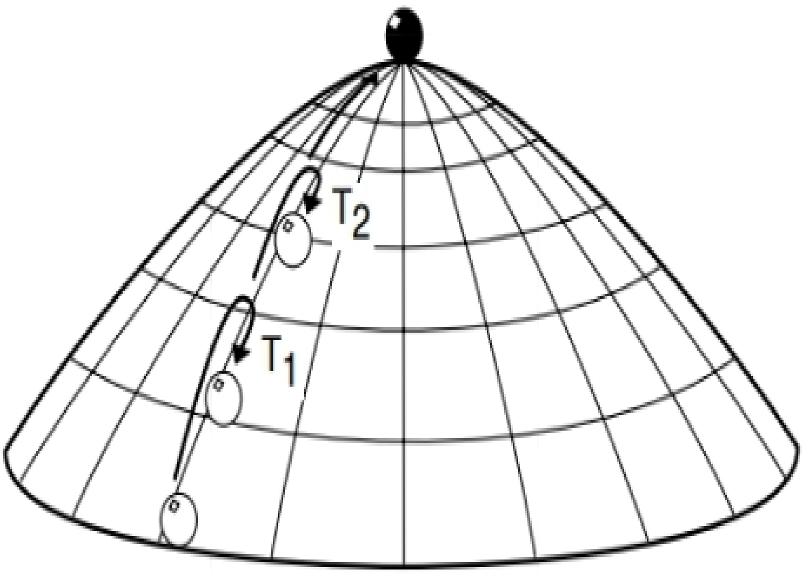

Figure 5: time reversal of the ball's spontaneous trajectory (Norton, 2003)

However, as one saw above, this would be forgetting that the solution obtained by time inversion is not the only trajectory starting from static conditions but, after analysis of the rotational symmetry of the problem, one among an infinity of simultaneous trajectories covering the surface of the dome.

Certainly, only one trajectory starting from the top will arrive at the bottom with the velocity vector in the exact opposite direction to that of the initial projection experiment but, without this arbitrary "final condition", nothing will force the static particle at the top to take this one direction rather than another (an infinity of others...).

In other words, a ball's trajectory going up the slope of the dome and stopping exactly at the summit is not time-reversible: if ***s(t)*** is a well-defined solution to the differential equation of dynamics, then ***s(-t)*** is no longer necessarily so. This is perhaps the first example of an irreversible law of classical physics where only fundamental forces are involved (without statistical, macroscopic, or empirical considerations, as with viscosity forces).

2.3 Ubiquity method and physical interpretation

It is now possible to outline a general method that would extend the principle of inertia to non-Lipschitz cases by demonstrating the ubiquity of any other differentiable solution to the differential equations of dynamics.

Indeed, from a classical physics perspective (excluding relativity and quantum mechanics), it is reasonable to eliminate any behavior of a rigid body that, in the same frame of reference, simultaneously adopts several mechanical behaviors. Mathematically, this is equivalent to the obvious reflex of searching for solutions to the motion among well-defined functions f, which associate at most one image $f(t)$ to each antecedent t.

From this perspective, if it is possible to immerse the problem in question, with zero initial conditions (such as a ball at the apex of a planar half-profile of the Norton dome), in a sufficiently large space where at least one ubiquitous behavior of this system exists, then the system will be declared non-physical or inconsistent. In other words, simply changing the immediate environment of a perfectly at-rest system (zero velocity, forces, and acceleration) without any dynamic influence is not enough to disturb it.

Depending on its nature, the system's expansion can generally occur either symmetrically (S) (for example, along an axis of rotation) or asymmetrically (A) (when, e.g., the initial state of rest on the right can be extended to the left). The immersion of force fields in larger systems should be inconceivable only in exceptional cases, such as cosmological models relating to the size and nature of physical space.

Subject to these assumptions, the *principle of inertia*, in its formulation by Newton (1687), states:

> *Corpus omne perseverare in statu sue quiescent vela movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.*

Would tend to become a theorem—**generalized here to rotating bodies**—even in cases where it has not yet been proven (namely, non-Lipschitz problems involving objects initially at rest):

> *[In any continuous and locally expandable force field], every coherent body maintains its state of rest or uniform rectilinear motion* ***[resp. uniform rotational motion]*** *unless it is forced to change state by [resultant] external forces* ***[resp. moments of force]*** *acting upon it.*

It would then no longer be necessary to artificially distinguish between Newtonian and non-Newtonian systems (Fletcher, 2012), Lipschitz and non-Lipschitz forces, physical and non-physical idealizations, etc.

The first fundamental classical principle capable of replacing the law of inertia, now a theorem, would then be the following:

Principle of prohibition of ubiquitous systems

(or principle of non-ubiquity, or consistency of dynamic trajectories…):

No behavior of a rigid dynamic system can be ubiquitous, i.e., extend over at least one spatial dimension in incompatible directions.

Two or more directions or trajectories followed by a moving object in the same frame of reference are simply said to be *incompatible* if they start from the same point at $t = 0$ and then diverge in space. Not only rectilinear motion but also rotations and rotating systems are permitted, as illustrated in case n°1 below (inverted pencil), where the dimension considered is an angle of rotation θ. Dynamical systems are said to be *ubiquitous systems* if they do not satisfy the principle of material consistency stated above.

Once again, spontaneous departures from the rest state without an initial external force are eliminated under these minimal assumptions. This definition of consistency aims to reject any behavior of an initially resting object that could be modified solely by the artificial extension of its space of study within the same frame of reference. Let us now turn to the application of this method to several historical problems.

3. Application to historical cases

3.1 Lipschitzian textbook case

The ubiquity method also solves cases involving Lipschitzian forces with zero initial conditions, without resorting to the Cauchy-Lipschitz theorem, nor performing a direct solution of the differential equation of dynamics.

Consider for example the well-known case of *a pencil balanced on its tip* (Synge & Griffith, 1959). The equation (25) of its angular motion on one half of its profile (e.g. with positive angles to the right) is derived from the principle of conservation of angular momentum:

$$\frac{d^2\theta(t)}{dt^2} = \omega^2 . sin\big(\theta(t)\big) \qquad (25)$$

where θ is the angle relative to the vertical, $0 \leq \theta \leq \pi/2$, and the impulsion (26):

$$\omega = \sqrt{\frac{mgl}{I}} \qquad (26)$$

With I the moment of inertia of the pencil around its tip, m the mass, l is the length (distance from the center of mass to the tip) and g is the acceleration due to gravity.

Initial conditions (27):

$$\theta(0) = 0, \qquad \frac{d\theta}{dt}(0) = 0, \qquad \frac{d^2\theta}{dt^2}(0) = 0 \qquad (27)$$

Since the force field in: $\theta \mapsto \sin\theta$ is Lipschitzian (its derivative with respect to θ, i.e. $\theta \mapsto cos\ \theta$, exists and is bounded) and $\theta = 0$ is a solution of the above equation, it is the only solution according to the Cauchy-Lipschitz uniqueness theorem.

Another method is to directly address the differential equation. For small angles θ, physicists use the approximation (28):

$$sin\ \theta \approx \theta \qquad (28)$$

Then they solve (29):

$$\frac{d^2\theta(t)}{dt^2} \approx \omega^2 . \theta(t) \qquad (29)$$

Solutions (30)-(31) are:

$$\theta(t) = A . e^{\omega t} + B . e^{-\omega t} \qquad (30)$$

$$\frac{d\theta(t)}{dt} = \omega . A . e^{\omega t} - \omega . B . e^{-\omega t} \qquad (31)$$

Thus, after applying the initial conditions (27) above:

$$A = 0, \qquad B = 0$$

Which leads to the unique zero solution (32):

$$\text{for all } \boldsymbol{t},\ \theta(t) = 0 \qquad (32)$$

The ubiquity method is much more straightforward. The equation (33) on the negative left half-plane is obtained for $-\frac{\pi}{2} \leq \theta \leq 0$:

$$\frac{d^2\theta}{dt^2} = -\omega^2 . sin(-\theta) \qquad (33)$$

Combining both positive and negative cases (25) and (33) leads to (34):

$$\frac{d^2|\theta|}{dt^2} = \omega^2 . sin|\theta| \qquad (34)$$

One then obtains a mathematically ill-defined solution for any non-zero solution of θ, given the null initial conditions, thus the constant rest: $\theta = 0$. Here, there is no need for the Cauchy-Lipschitz uniqueness theorem, sine approximation, equation solving, or coefficient determination. Let one clarify that the dynamic equation obtained using the absolute value of the target variable remains the same regardless of the initial conditions.

For example, if $\theta(0) \neq 0$ or $d\theta/dt \neq 0$ (broken symmetry in initial conditions), then $|\theta|$ will still yield two branches for θ, but one of these branches will necessarily be eliminated by the initial data; this is no longer possible if the latter are equal to zero: only a breaking of symmetry in the initial conditions would prohibit the existence of a solution branch of the motion on the other half-profile, thus eliminating the ubiquity automatically generated in the system by the absolute value in the differential equation.

Besides, the Cauchy-Lipschitz Theorem doesn't work if the force field (here $F(\theta)$) is not defined on a neighborhood, i.e. a set containing an open interval, of the initial conditions (here θ_0), rendering the theorem inoperative, when the existence of a unique solution on the half-profile can be demonstrated by other means (such as using elementary integration).

For example, studying the pencil movement on the half-plane $\theta \geq 0$ for $\theta(0) = 0$ and initial speed $d\theta/dt(0) < 0$ doesn't even guarantee the existence of a solution at $t > 0$ on this right half-plane unless one extends $F(\theta)$ on some open negative interval of θ containing zero. This remark justifies one's decision to henceforth avoid using the Cauchy-Lipschitz theorem in the study of systems defined on half-planes.

As this case of the inverted pencil shows, the consideration of geodetic coordinates such as rotation angles is allowed in the ubiquity method: it then generalizes to the conservation of the angular momentum of an isolated rotating system, or of a system initially at rest, even in a field of non-Lipschitz dynamic moments.

3.2 Non-Lipschitzian cases

One is going now to analyze in depth and develop six historical non-Lipschitzian cases (extracted in particular from the work of M. Van Strien, 2014, pp. 32-45) through the prism of the ubiquity method proposed in this work. The method followed here systematically consists of immersing each system studied in a larger space where it will be possible to resolve the differential equation of the dynamics in several directions and bring about a contradiction in the behavior of the mobile in order to keep the only solution of constant rest.

Poisson (1806) — Force $F(r) = c.r^a$ with $0 < a < 1$

Poisson seems to be historically the first scholar (Poisson, S.-D. (1806). "Mémoire sur les solutions particulières des équations différentielles et des équations aux différences". *Journal de l'École Polytechnique*, 6(13), 60-125, cited in van Strien, 2014) to study multiples solutions of Newtonian differential equations, leaving open or not the possibility that they could be encountered in the physical world. He studied non-Lipschitzian forces before its time (the Cauchy-Lipschitz theorem would only begin to emerge decades later...).

Starting from the differential equation of a classical unit mass particle in a rectilinear motion subject to an accelerating force $F(r) = c.r^a$, with r the distance from the origin, a and c constants, $0 < a < 1$, he gets the differential equation (35):

$$\frac{d^2r}{dt^2} = c.r^a \qquad (35)$$

At zero initial conditions for position and velocity: $r(0) = 0, r'(0) = 0$.

This equation admits both the trivial static solution (36):

$$\forall t, \quad r(t) = 0 \qquad (36)$$

And a family (37) of spontaneous motion solutions for arbitrary time T:

$$r(t) = A(t-T)^{\frac{2}{(1-a)}} \text{ for } t \geq T \qquad (37)$$

With A constant.

First, one could consider a circular symmetry, as for the Norton's dome, or just a mirror symmetry to complete the real axis for the coordinate r in one dimension. Then, by extending to a signed coordinate s, the equation becomes (38)-(39):

- For $s \geqslant 0$**:**

$$\frac{d^2 s(t)}{dt^2} = c \cdot s^a(t) \qquad (38)$$

- For $s \leqslant 0$**:**

$$\frac{d^2 s(t)}{dt^2} = -c \cdot \left(-s(t)\right)^a \qquad (39)$$

Thus, general equation (40), after summarization using $s \to |s|$ **:**

$$|\ddot{s}|(t) = c \cdot |s(t)|^a \qquad (40)$$

From which one obtains both solutions $s(t)$ and $-s(t)$ as valid, leading to spatial ubiquity, i.e. simultaneous departures in opposite directions.

But, even without summarization, one shows a half-profile contradiction: applying Newton's laws on a left domain with purely vertical force (no tangential field ⇒ Lipschitzian force with respect to the variable$s \leqslant 0$) yields persistence at rest as the *unique solution* (41) at left side–which is nothing other than the solution of inertia principle:

$$d^2 s/dt^2 = 0 \Longrightarrow \forall t, s(t) = 0 \qquad (41)$$

While the right-side solution permits spontaneous departure. This contradiction cannot be resolved without imposing an asymmetric/directional force.

A single rigid object governed by Newtonian laws cannot behave differently across adjacent domains: the model is thus physically inconsistent, but the hidden contradiction only appears by geometrically extending the space of the system.

Poisson (1806): friction-like Force $\boldsymbol{F(u) = -d\sqrt{u}}$

A unit-mass particle is subject to a non-conservative force (42) that depends here on speed $\boldsymbol{u}$, not position (van Strien, 2014):

$$F(u) = -d\sqrt{u} \qquad (42)$$

$$with\; u \geq 0,\; \boldsymbol{d} \text{ a constant}$$

which leads to the differential equation (43) of movement:

$$\ddot{r}(t) = -d\sqrt{\dot{r}(t)} \qquad (43)$$

$$r(0) = 0, \dot{r}(0) = 0.$$

Depending on the sign of $\boldsymbol{d}$, this applied force is oriented along the velocity vector (driving force) or opposite (braking force). Again, a static solution:

$$r(t) = 0, \forall t.$$

By posing (44):

$$u(t) = \dot{r}(t) \qquad (44)$$

This equation becomes (45):

$$\dot{u}(t) = -d\sqrt{u(t)} \qquad (45)$$

If $d \leqslant 0$ and $t \geqslant T$:

Integration (46) gives:

$$\int du/\sqrt{u} = -d\int dt \qquad (46)$$

$$2\sqrt{u} = -d(t-T)$$
$$u(t) = \frac{d^2}{4}(t-T)^2$$

One gets non-trivial solutions (47)-(48) for $t \geq T$, with arbitrary time departure T:

$$u(t) = \dot{r}(t) = \frac{d^2}{4}(T-t)^2 \qquad (47)$$

$$r(t) = \frac{d^2}{12}(t-T)^3 \qquad (48)$$

Velocity is increasing from $u(T) = 0$ to $u(t) > 0$.

But, for $d \geqslant 0$, one gets after integration (49) a field of frictional (decelerating) forces. If $u(T) = K > 0$, $t \geqslant T$ and $t \leqslant T + \frac{2\sqrt{K}}{d}$:

$$\int \frac{du}{\sqrt{u}} = -d\int dt \qquad (49)$$

$$2\sqrt{u} - 2\sqrt{K} = -d(t-T)$$

$$u(t) = \left(\sqrt{K} - \frac{d}{2}(t-T)\right)^2$$

Non-trivial solutions (50)-(51) are then decelerating movements from $r(T) = R$ to $r = R + \frac{2}{3d}\sqrt{K}$:

$$u(t) = \dot{r}(t) = \left[\sqrt{K} - \frac{d}{2}(t-T)\right]^2 \qquad (50)$$

$$r(t) = R + \frac{2}{3d}K\sqrt{K} - \frac{2}{3d}\left[\sqrt{K} - \frac{d}{2}(t-T)\right]^3 \qquad (51)$$

By modifying *d*, one obtains trajectories with spontaneous motions, braking, stops or restartings. As force is only studied by Poisson for $u(t) = \dot{r}(t) \geq 0$, one can extend it to the whole real axis of speeds by considering (with variable *v* the signed version of velocity *u*):

- $\boldsymbol{v \geqslant 0}$ (the particle moves towards the $s > 0$) obeys to (52):

$$\dot{v}(t) = -d \cdot \sqrt{v(t)} \qquad (52)$$

- $\boldsymbol{v \leqslant 0}$ (the particle moves towards the $s < 0$) obeys to (53):

$$\dot{v}(t) = +d \cdot \sqrt{-v(t)} \qquad (53)$$

Then (54), joining both cases:

$$|\dot{v}|(t) = -d \cdot \sqrt{|v(t)|} \qquad (54)$$

If d $\leqslant 0$ for example (driving force), solutions become (55)-(56), for $t \geqslant T$:

- on the semi-axis $v > 0$:

$$v_+(t) = +\frac{d^2}{4}(T-t)^2 \qquad (55)$$

- on the semi-axis $v < 0$:

$$v_-(t) = -\frac{d^2}{4}(T-t)^2 \qquad (56)$$

Trajectories (57)-(58) are then:

- on the semi-axis $s > 0$:

$$s_+(t) = +\frac{d^2}{12}(t-T)^3 \qquad (57)$$

- on the semi-axis $s < 0$:

$$s_-(t) = -\frac{d^2}{12}(t-T)^3 \qquad (58)$$

Here appears the same contradictions as above: once symmetrized, the system allows arbitrary redirections in velocity space, then in positions, revealing physical inconsistency. Same thing on a asymmetrized half-profile: right-side motion coexists with inert left-side rest, violating object uniqueness. Again, ubiquity of motion appears, in contradiction with Newtonian realism.

Poisson didn't push his analysis to a larger physical space; however, he discusses at length the relevance of singular solutions (van Strien, 2014). For him, as for Duhamel later (see below), it is clear that if one has to choose, then one must eliminate the solutions that do not respect the principle of inertia. Poisson seems to interpret the latter as the attribution of causality to the notion of force.

Yet, as one will see with Duhamel, nothing in the “indeterministic” dynamical equations in themselves allows to decide between the constant rest and the regular “acausal” solutions. The preference they give to the solution of rest by the principle of inertia looks more like a metaphysical than a mathematical choice, strictly speaking.

Duhamel (1845) — Philosophical objection to non-uniqueness

This case (Duhamel, 1845, *Cours de mécanique de l’école polytechnique*, vol. 1, pp. 265. Paris: Bachelier, cited in van Strien, 2014) is more modest in its mathematical scope, but interesting from a doctrinal point of view, showing how some physicists recognized the existence of multiple solutions, while explicitly choosing to reject them on the basis of physical, metaphysical or moral reasoning (van Strien, 2014). Duhamel revisits the examples studied by Poisson, notably those where the force is:

$$F(v) = -c \cdot v^{a}, \qquad 0 < a < 1$$

He does not present a new system, but generalizes those of Poisson (where $a = \frac{1}{2}$). Considering the equation (59) of motion:

$$\ddot{r}(t) = -c \cdot \dot{r}^{a}(t) \qquad (59)$$

Duhamel mathematically admits that there are an infinite number of solutions, as seen previously, but he maintains on physical grounds that only the rest solution (60) is acceptable:

$$r(t) = 0, \qquad \forall t \qquad (60)$$

Although he does not formalize this idea, Duhamel seems to invoke the principles of inertia and causality to reject indeterminism and disqualify uncaused solutions. However, as already seen earlier with the Norton's dome and Poisson, the principle of inertia is not independent from the dynamics equation: in non-Lipschitzian cases, permanent rest and delayed starts are both exact mathematical solutions of the equation of motion. Inertia is just a particular one, unable *per se* to discriminate among others or declare any non-static solution unphysical.

Since there is no internal criterion in the three classical laws of motion to choose between indeterministic solutions, one extends the differential equation (61) in $\boldsymbol{r}$ to the symmetrized system in $\boldsymbol{s}$:

$$|\ddot{s}|(t) = c \cdot |s(t)|^{a} \qquad (61)$$

This equation again allows for dynamical ubiquity which:

- Violates the physical identity of the particle;
- and logically invalidates solutions other than rest.

Then, one’s method provides a logical formalization of what Duhamel intuitively asserts: multiple solutions lead to a global spatial contradiction. One preserves the entire Newtonian framework: no additive principle, but a global logical test: if the set of solutions creates a spatial contradiction (ubiquity), then they must be eliminated. This principle is not dynamic, but logicogeometric. It allows one to decide between solutions without betraying the initial equations or importing a heavy external axiom (such as a minimization or stability principle).

Finally, to answer to Duhamel, inertia in itself cannot rule out others solutions by fiat. Perhaps this is why, aware of its weaknesses, scientists like Newton elevated it to the rank of principle. Instead, ubiquity of undeformable objects can more rigorously justify the rejection of spontaneous solutions without overdetermining classical physics. It makes uniqueness not arbitrary, but necessary for the coherence of the physical world: one object equals one position at each instant.

As in the previous section, one will propose abandoning inertia as a simple principle, subject to the arbitrariness of physical systems, and making it more a "theorem". Move on to the next historical cases that will only confirm this recurring need to evolve the scientific paradigms.

Boussinesq (1879) — Generalized dome

Boussinesq discusses the same kind of bifurcation in mechanical systems and introduces the idea of spontaneous rupture beyond deterministic prediction. He writes that when the equations of mechanics are no longer sufficient to determine the motion, it is necessary to invoke an external cause, which he calls *agent directeur*, i.e. "directing agent" (van Strien, 2014).

Boussinesq builds a mechanical system specifically intended to highlight non-unique solutions linked to the failure of the Lipschitz condition. He designs a surface of revolution (dome) on which a particle of unit mass is deposited, subject only to gravity (van Strien, 2014, p. 38).

This dome is defined by a half-profile on the plane, with the height as a function of the path $r \geq 0$ from the apex. Boussinesq generalizes the classical form by proposing (62):

$$h(r) = \frac{K^2}{2g}\left(log\left(\frac{a}{r}\right)\right)^{2k} r^{2m} \qquad (62)$$

One might recover the Norton's Dome by setting $a = e.r$ (where $\boldsymbol{e}$ is the Euler constant), $m = 3/4$, $K^2 = 1/m$ (van Strien, 2014).

A particle slides on the surface described by this height function which becomes non-Lipschitz at $r = 0$ for $1/2 < m < 1$. The movement is governed by the law (63)-(64):

$$\ddot{r}(t) = g\frac{dh}{dr} \qquad (63)$$

$$\frac{dh}{dr} = \frac{K^2}{g}\left[m\left(log\left(\frac{a}{r}\right)\right)^{2k} - k\left(log\left(\frac{a}{r}\right)\right)^{2k-1}\right]r^{2m-1} \qquad (64)$$

Both solutions of continuous rest (65) and departure are mathematically permitted:

$$r(t) = 0, \forall t \qquad (65)$$

and an infinity of solutions (66) of type:

$$r(t) = f(t - T), \forall t \geqslant T \qquad (66)$$

with arbitrary $\boldsymbol{T}$ and: $f(0) = f'(0) = 0$

Boussinesq deduces that, without external cause, the system is unable to choose "by itself" one direction of movement, which he philosophically interprets as the introduction of a non-material *guiding principle* (idem). Applied to biology, he identifies this hidden variable with free will (Boussinesq, 1877).

Now, let one symmetrically extend this half-profile around the dome axis by mean of the signed coordinate s. Like in Norton's dome, the Boussinesq profile and its associated differential equation become (67)-(69):

$$H(s) = h(|s|) = \frac{K^2}{2g}\left(log\left(\frac{a}{|s|}\right)\right)^{2k} |s|^{2m} \qquad (67)$$

- For $s \geqslant 0$: $\ddot{s}(t) = g \cdot \frac{d}{ds}H(s) = g \cdot \frac{d}{ds}h(s)$ (68)

- For $s \leqslant 0$: $\ddot{s}(t) = g \cdot \frac{d}{ds} H(s) = g \cdot \frac{d}{ds} h(-s) = -g \cdot \frac{d}{d(-s)} h(-s)$ (69)

Then (70), for all *s*:

$$|\ddot{s}|(t) = g \cdot \frac{dh(|s|)}{d|s|} \qquad (70)$$

which releases the new simultaneous solutions (71) for t ⩾ T:

$$|s(t)| = f(t - T) \qquad (71)$$

then:

- Solution (72) on the semi-axis $s > 0$:

$$s_+(t) = f(t - T) \qquad (72)$$

- Solution (73) the semi-axis $s < 0$:

$$s_-(t) = -f(t - T) \qquad (73)$$

Extension to negative *s* and transformation to $|s|$ allows incompatible mirrored solutions. At any time, *T*, the object can go both in one direction and in the opposite (geometrical ubiquity). Only the standing rest state solution respects the principles of material identity, inertia and spatial coherence.

Again, physical contradictions arise in an asymmetrized half-profile from:

– Left side: vertical forces only (if no tangential force fields) → rest enforced on this side by lack of tangential field.
– Right side: motion predicted → contradiction unless a directional trigger is introduced to break the inertia.

Boussinesq (1879) — Two particles in indeterministic interaction (Boscovich atomic model)

Here, Boussinesq describes a two-body system (sometimes called "atoms") subject to a non-Newtonian central force (van Strien, 2014, p. 40):

– One of the examples considered seems to be inspired by the Boscovich's atomistic theory of matter (1758, cited in idem) developed in his *Theoria philosophiae naturalis*, where atoms are not considered as extended indivisible bodies but as points and centers of force, spinning around each other: Boscovich believed that the force between atoms was repulsive at very short distances, attractive at macroscopic distances-in accordance with the law of universal gravitation-and changed sign (alternately attractive and repulsive) in the intermediate zone.
– The interaction potential is not regular at equilibrium distances R_i, but remains finite and continuous.
– Unstable circular orbits are possible, where the particles rotate around each other.

Indeed, by moving from its attractive zone to its opposite, this interaction should pass through a state of zero force. Boussinesq expects the singular orbits at such points. In this configuration, the two particles can remain in unstable relative equilibrium at distance *R* in a circular orbit, or begin to move spontaneously to another distance at an indeterminate time *T*, "without cause".

The fundamental equation of dynamics (74) for the distance *r* between the two particles is:

$$m\ddot{r} = F(r) \qquad (74)$$

Boussinesq's thought can be mathematically translated by the following potential *V* (75) from which one derives the corresponding atomic interaction force (76):

$$V(r) = -C \cdot |r - R|^{1+a} \qquad (75)$$

$$C > 0, \qquad 0 < a < 1$$

$$\Longrightarrow F(r) = C(1 + a) \cdot sgn(r - R) \cdot |r - R|^{a} \qquad (76)$$

So, dynamics equation (77) (defining constant $k = C(1 + a)$):

$$\ddot{r} = k \cdot sgn(r - R) \cdot |r - R|^a \qquad (77)$$

N.B. let one point out that this formulation of atomic interaction respects the Boscovich conditions around unstable equilibrium points ($F(r) > 0 \; if \; r > R$ and $F(r) < 0 \; if \; r < R$). Thus, one finds:

- Trivial solution (78):

$$r(t) = R, \qquad \forall t \qquad (78)$$

- Non-trivial solutions (79):

$$r(t) = R \pm A(t - T)^{\frac{2}{1-a}}, \qquad for \; t \geq T \qquad (79)$$

The two particles could either stay relatively at rest, or spontaneously move closer or further apart–they only oscillate around stable equilibrium points. These solutions appear for the same initial conditions and Boussinesq treats them as alternative and mutually exclusive solutions rather than simultaneous and contradictory.

Things will get more obvious by introducing the signed variable $s = r - R$ which is equivalent at first to what Boussinesq did, but one will pay more attention in interpreting his solutions. The general equation becomes (80):

$$|\ddot{s}|(t) = k \cdot |s(t)|^a \qquad (80)$$

Hence the following possible non-trivial solutions (81) for all ***T***:

$$|s(t)| = A(t - T)^{\frac{2}{1-a}}, \qquad for \; t \geq T \qquad (81)$$

$$with \; A = \left(\frac{2(1+a)}{k(1-a)^2}\right)^{\frac{1}{a-1}}$$

It admits both:

– Inward solutions (82) on the semi-axis ***s < 0***: $s_{-}(t) = -A(t - T)^{\frac{2}{1-a}}, \; for \; t \geq T$ (82)

– And outward solutions (83) on the semi-axis ***s > 0***: $s_{+}(t) = +A(t - T)^{\frac{2}{1-a}}, \; for \; t \geq T$ (83)

System allows motion in opposing directions from identical initial conditions. Unless a symmetry-breaking directional force is used, this modelling of free will in Boscovich atomic theory is then contradictory. As previously, it can be shown that the only consistent solution is perpetual relative rest of the two particles at distance R.

As one has mentioned, non-deterministic Norton-type solutions are actually irreversible in time. Besides, the problem of the arrow of time, in latent conflict with the reversibility of the laws of mechanics, plagued 19th-century physics in its construction of thermodynamics—until Boltzmann provided a statistical interpretation of entropy (van Strien, 2013).

However, Boltzmann's irreversibility is not strictly speaking true since the phenomena almost surely end up returning to their starting point after a long time, according to Poincaré's recurrence theorem (idem). For unit mass, a non-reversible trajectory is (84):

$$s(t) = \pm A(T - t)^{\frac{2}{1-a}}, \qquad for \; 0 \leq t \leq T \qquad (84)$$

Thus velocity (85) and acceleration (86):

$$v(t) = \pm A \left(\frac{2}{1-a}\right)(T - t)^{\frac{1+a}{1-a}} \qquad (85)$$

$$a(t) = \pm A \left(\frac{2(1+a)}{(1-a)^2}\right)(T - t)^{\left(\frac{2a}{1-a}\right)} \qquad (86)$$

One just obtains this trajectory by reversing in time the Boussinesq solutions. As expected, (84)-(86) verifies:

$$s(T) = 0,$$
$$v(T) = 0,$$
$$a(T) = 0.$$

Initial conditions (87)-(88) are written as follows:

$$s_0 = A.T^{\left(\frac{2}{1-a}\right)} \qquad (87)$$

$$v_0 = -A\left(\frac{2}{1-a}\right)T^{\left(\frac{1+a}{1-a}\right)} \qquad (88)$$

Then a simple relation (89) between them:

$$\frac{s_0}{v_0} = -\frac{(1-a)}{2}.T \qquad (89)$$

The critical velocity (90) required to fix the incident particle at *r = R* (i.e. *s=0*) is:

$$v_{cr} = -\sqrt{2C}.s_0^{\frac{1+a}{2}} \qquad (90)$$

From the theorem of kinetic energy, one derives (91):

$$\frac{1}{2}m.v(s)^2 - \frac{1}{2}m.v_0^2 = -\frac{1}{2}m.v_{cr}^2\left(1 - \left(\frac{s}{s_0}\right)^{1+a}\right) \qquad (91)$$

which reflects the relative loss of energy by one's particles as it moves against a force of repulsion towards the unstable equilibrium point until it reaches position ***s***. If its initial speed is insufficient, the particle will turn back (collision and rebound of particles against each other). Otherwise, for $v_0 > v_{cr}$, it will pass through the equilibrium point until it reaches the next center of repulsion, according to Boscovich's theory, with some residual kinetic energy, at speed ***v*** in (92) such that:

$$\frac{v(r=R)}{v_0} = \left(1 - \left(\frac{v_{cr}}{v_0}\right)^2\right)^{\frac{1}{2}} < 1 \qquad (92)$$

The case $v_0 = v_{cr}$ introduces, even for a single pair of particles, a temporal irreversibility at the very heart of Newtonian mechanics. Then, although now obsolete, non-Lipschitzian atomistic models such as Boscovich and Boussinesq's would allow the development of a primitive thermodynamics which would respect the physical arrow of time: two particles meeting could form in a finite time a system bound at a fixed equilibrium distance. With the center of inertia of these two linked-particles systems moving at a speed equal to the average of the two initial speeds, an explanation for thermalization between two bodies, the expansion of gases, and other irreversible natural phenomena, might be proposed.

Boussinesq's quest was not absurd: even today, modern research attempts to explain biology, human consciousness and certain neurological processes using the indeterminacy principle of quantum mechanics (see e.g. R. Penrose, 2022). However, the French scientist is aware of the rarity of these singularities: particles must be calibrated with precise initial conditions values in order to be able to fix themselves at unstable equilibrium points. He uses numerous arguments to justify the abundance of such centers of free will, at least in living matter (possessing a particular chemistry). The same effort should be made to explain the omnipresence of irreversible trajectories in inert matter.

Furthermore, nothing prevents molecular agitation from continuously breaking and reforming unstable bonds between atoms: the presence of free atoms would not, in theory, allow to unilaterally reconstruct the sequence of past events. Consequently, indeterminism, in the sense of the multiplicity of possible trajectories, could universally apply to matter. The passage of time is accompanied by a loss of past information and the physical impossibility of returning to previous states.

Stable equilibrium points (Lipschitzian or not) would be classically responsible for homogenizing macroscopic energy within the fluid, while the non-Lipschitzian points of unstable equilibrium, as one has seen, ensure the microscopic irreversibility of its thermodynamics through the continuous creation and destruction of unstable atomic pairs with lower kinetic energy and higher potential energy than stable particle pairs.

There are no corresponding trajectories that would lead the atom to the stable equilibrium points in a finite time: indeed, the time reversal of such trajectories would yield solutions that spontaneously diverge from these equilibrium points, contradicting their stability. Similarly, Lipschitzian forces cannot lead to such reversed trajectories, according to the Picard-Lindelöf uniqueness theorem. Non-Lipschitzianity and instability would thus become an essential characteristic of irreversibility. Here, the entropy $\boldsymbol{S}$ would no longer be associated, as in Boltzmann, with the number $\boldsymbol{\Omega}$ of possible microscopic states of a system per macroscopic state ($S = k_B ln\, \Omega$), but with the quantity, the influence, or the occupancy rate, like so many ticks of a universal clock, of non-Lipschitzian equilibrium centers in atomic force fields.

Those last points could prove to be more abundant than expected in Nature. To take this into account, it would be necessary to correct the Lipschitzian interaction forces of the Coulomb or Van der Waals type by non-Lipschitzian terms valid at certain distances from the atom. There is nothing, in principle, to preclude the existence of a kinetic temperature T_c in this model, analogous to Boltzmann's temperature ($< v^2 > \propto T_c$): certain values of this temperature would favor the proportion of atoms that will reach the critical velocity v_{cr} at collision-free intermolecular distances from the target particles.

Boussinesq was perfectly aware of the necessary adjustment of certain physical parameters (such as the initial relative velocity of colliding particles) to make his model more realistic and to favor the regular occurrence of unstable non-Lipschitz equilibrium points (Van Strien 2014, pp. 40–41). He even went so far as to quantify their theoretical non-zero probabilities of appearance in matter. These advances naturally benefit a thermodynamic interpretation of his atomic model, where the frequencies of free acts become frequencies of irreversible processes.

His atomistic theory was unintentionally more of a simplified classical thermodynamic model than a physical model of free will. Strictly speaking, there are no more thermodynamics of this kind in nature than there are Norton domes, but their existence in classical physics seems quite real. To make it physical would presuppose the discovery of new fundamental non-Lipschitz interaction laws between molecules. Anyway, by seeking the origin of free will (Müller, 2015) Boussinesq could have unwillingly opened a door to the fatalism of a time always flowing in the same direction.

Bertrand (1878-79)—Rejection of Boussinesq's framework and Philosophical Critique of Mechanical Indeterminism

In this last case, Joseph Bertrand rejects any physical allowance of non-uniqueness, demanding a 'true' hidden law. He reacted to Boussinesq's proposals, particularly the idea that certain mechanical equations might not unequivocally determine the evolution of a system (Bertrand, J. (1878). *Compte rendu de « Accord des lois de la mécanique avec la liberté de l'homme dans son action sur la matière » de J. Boussinesq*. Journal des Savants, septembre, 517–523, cited in van Strien, 2014). He directly criticized:

- The idea of a "guiding principle" (in Boussinesq's sense), that is some effect of the mind on the matter without any helping force.

– The very physical existence of systems with multiple spontaneous solutions.

His central postulate may be stated as: if an equation admits multiple solutions from the same initial conditions, it does not correctly reflect the real physical nature. Bertrand suspected discontinuous or hidden laws may underlie the apparent indeterminism. He then suggested that the true law of nature was different:

- Perhaps more disjoint,
- Perhaps non-continuous,
- Perhaps involving jumps in behavior. He therefore refuses to accept the idea that nature itself would allow multiple trajectories compatible with the same initial state.

For him, the plurality of solutions is not a real paradox, but rather evidence of an error in the modeling (see idem). Anyway, as in the previous cases, Bertrand's point of view could now be formally assessed. By requiring a classical directed force to break inertia, without needing to invent an extra-physical guiding principle, one’s approach should allow for a rigorous justification of Bertrand's rejection of indeterminism as well as of any “invisible” (but unspecified) principle of determination.

Yet, non-Lipschitz equations are not necessarily “wrong”, “unphysical” or “incomplete”: on the contrary, they successfully reveal their bad solutions as totally contradictory, in the sense of leading to mathematically ill-defined applications and physically incompatible predictions. The very structure of the trajectories is enough to decide. Multiple solutions, except zero motion (constant rest), lead to geometric contradiction without recourse to hidden variables, bad modeling or unknown laws.

4. Conclusion

In summary, there appears at first glance to be a significant flaw in the axiomatic system of the three laws of Newtonian physics. Problems such as Norton's dome or revisited historical cases from the 19th century all converge on the idea of a fundamental indeterminacy between the consequences of the first law of motion (the principle of inertia) and the second (the general equation of motion), without the third law of action/reaction being able to resolve them.

The necessary arbiter for this dispute would rather be found in a "postulate of classical non-ubiquity," the physical equivalent of the law of excluded middle: a point mass or a rigid object can move in one direction or the other, but not in both simultaneously. Any motion that appears locally coherent but behaves differently in the same frame of reference, simply by broadening its scope without altering anything or by changing the geographical perspective, should be rejected as non-physical. This is merely a moderate form of local realism in classical (non-quantum) physics. Mathematically, this postulate amounts to searching for well-defined functions as non-multivalued solutions to the fundamental differential equation of dynamics.

By confirming the principle of inertia as the "winning" mathematical solution against its rivals of indeterminism, an argument like that of the ubiquity of spontaneous solutions can prove decisive in resolving these age-old conflicts concerning the laws of motion, dating back to at least 1806. It makes the principle of inertia (seen as conservation of linear momentum or conservation of angular momentum of rotating systems) a kind of Cauchy-Lipschitz theorem generalized to non-Lipschitz cases.

Furthermore, the study of the Norton dome and other historical cases has confirmed that one can no longer consider all the fundamental laws of mechanics as reversible over time, in the sense of the irreversibility of solutions to differential equations. There appears to be a close correlation between the behavior of non-Lipschitz forces, unstable equilibria, and processes characterized by an arrow of time. The principle of inertia itself seems to have always been, in essence, a second law of thermodynamics.

At the same time as spiritualism emerged in the heart of the rationalist 19th century, "magical" solutions for spontaneous motion appeared amidst the rigid differential calculus of Newtonian determinism. These non-Lipschitz oddities ignited heated worldwide debates on indeterminism, the existence of free will, the power of mind over matter, human moral freedom, and so on. In the 19th century, those discussions were mainly centered in France, but it also actively involved famous foreign authors such as Maxwell, Kelvin, Du Bois-Reymond... The controversy resurfaced on a much more international scale in the late 20th/early 21st century. The study of several of the physical problems fueling these debates seems to have revealed their true nature in light of the recent Norton dome paradox.

It might be argued that one has merely shifted the problem from one “paranormal phenomenon” to another, that one refutes “telekinesis” by denying the existence of “ubiquity”, that one presumes the "Doppelgänger" to be more impossible than the “spontaneous movements” of inert objects. However, one demonstrates more precisely that these seemingly causeless movements of matter were in reality nothing more than disguised ubiquitous behaviors. It remains for us to provide the principle of inertia with a rigorous mathematical demonstration in its most general form, including the equivalence principle of special and general relativity in space-time. A door is also left open to quantum mechanics, which, as with the Norton dome, might present only a deceptive facade of indeterminacy. All these points are the subject of ongoing development.

Here, the main question will arise of the risk of basing entire philosophies on opaque and potentially erroneous scientific calculations, just as in the past (when philosophy was predominant over science), the risk was conversely of basing scientific theories too heavily on abstract philosophical dogmas. Thus, rather than a new "Notre-Dame" of philosophical indeterminism, Norton's dome would suddenly reveal itself as yet another cathedral of scientific determinism.

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